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# CURVES WITH A DIRECTRIX

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DISSERTATION

Submitted to the Board of University Studies of the Johns Hopkins University  
in conformity with the requirements for the degree of  
Doctor of Philosophy

BY

CLYDE SHEPHERD ATCHISON

1907

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To Professor Morley, without whose helpful suggestions and constant inspiration this paper would have been impossible, and to Dr. Cohen, Dr. Hulbert, and Dr. Coble for their encouragement in his university course, the author desires thus to express his thanks.



## *Curves with a Directrix.*

BY CLYDE S. ATCHISON.

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### *Introductory.*

It is a well-known fact of elementary geometry that the orthocentre of any three lines of a parabola, which is of course uniquely determined by four lines, lies on a line, called the directrix of the parabola. In his *Orthocentric Properties of the Plane  $n$ -Line*, Professor MORLEY\* has extended the meaning of the term orthocentre, so that with every odd number of lines there is associated an orthocentre, and with every even number of lines a directrix. It is my purpose to discuss the class of curves, of which the orthocentre of a certain number of lines shall always lie on a line, the directrix of the curve in question. As the curves to be considered are entirely rational, throughout the discussion, I shall employ almost exclusively the method of vector analysis, worked out by Professor MORLEY, which is especially convenient for the purpose. The line equation of a curve is here uniformly expressed by means of conjugate coordinates†, while the point equation is expressed by means of what is commonly called the map equation; that is, a point of the curve is expressed as a rational algebraic function of a parameter  $t$ , which is limited to the unit circle, and is, therefore, always equal in absolute value to unity. Throughout  $y$  and  $b$  will be considered as conjugates of  $x$  and  $a$  respectively; that is, if  $X$  and  $Y$  are rectangular coordinates of a point, then

$$\begin{aligned}x &= X + iY, \\ y &= X - iY,\end{aligned}$$

and similarly for  $a$  and  $b$ .‡

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\* *Transactions of the American Mathematical Society*, Vol. IV (1903), pp. 1-12.

† F. FRANKLIN: Some Applications of Circular Coördinates, *American Journal of Mathematics*, Vol. XII (1890), p. 161; F. MORLEY: On the Metric Geometry of the Plane  $n$ -Line, *Transactions of the American Society*, Vol. I (1900), p. 97.

‡ H. A. CONVERSE: *Annals of Mathematics*, Ser. II, Vol. V (1904), pp. 106-109, where is given a full explanation of these methods.

Employing the notation used in the memoirs of Prof. MORLEY, a line  $\alpha$  is written in the form

$$x t_\alpha + y = x_\alpha t_\alpha = y_\alpha,$$

where  $x_\alpha$  is the reflection of the origin in the line, and  $-1/t_\alpha$  is the clinant.

The definition of the first orthocentre, as extended for  $2n-1$  lines, is the point of intersection of the  $2n-1$  perpendiculars, let fall from the centres of the  $2n-1$   $\Delta^{2n-3}$ 's, one of which is uniquely determined by every  $2n-2$  lines, upon the line left out in each case. By a\*  $\Delta^{2n-3}$  is meant a deltoid of order  $2n-2$  and class  $2n-3$ , having  $2n-3$  cusps, and containing the line at infinity as a special  $(2n-4)$ -fold line, being tangent to it  $n-2$  times at  $I$  and  $n-2$  times at  $J$ . Thus, in the case of three lines, the three  $\Delta$ 's each of which is determined by two lines of the three, are simply points, the vertices of the triangle of the three lines, and the orthocentre is the intersection of the perpendiculars dropped from the vertices upon the opposite sides. But in the case of five lines, every four lines uniquely determine a  $\Delta^3$ , and the perpendiculars dropped from the centres of these five  $\Delta^3$ 's upon the line omitted in each case meet in a point, which is the orthocentre of the five given lines.

For  $2n-1$  lines, the first orthocentre\* is of the form

$$p_{n-1} = a_1 - S_1 a_2 + S_2 a_3 - \dots + (-)^n (\dots + S_{n-2} a_{n-1} - S_{n-1} a_n),$$

where  $S_i$  stands for symmetric functions of the  $2n-1$   $t$ 's, and the  $a_\alpha$  are the so-called characteristic constants of the lines, and in the case of  $2n-1$  lines

$$a_\alpha = \sum^{2n-1} \frac{x_1 t_1^{2n-1-\alpha}}{(t_1-t_2)(t_1-t_3)\dots(t_1-t_{2n-1})}.$$

From the results deduced in the *Orthocentric Properties*, the equation of the directrix for any even number of lines,  $2n$ , may readily be written :

$$\frac{x - p_{n-1}}{a_{n+1}} - \frac{y - q_{n-1}}{a_n} = -S_n,$$

where  $p_{n-1}$  is a point of the form of the orthocentre of  $2n-1$  lines, but constructed for a  $2n$  line, and  $q_{n-1}$  is its conjugate. It is evident, therefore, that the clinant of the directrix of  $2n$  lines is

$$\frac{a_{n+1}}{a_n}.$$

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\* *Orthocentric Properties of the Plane n-Line*, p. 5.

## § 1. Equation of the Curve.

Suppose any even number of lines are given,  $2n$ . They have a fixed directrix, whose clinant is  $\frac{a_{n+1}}{a_n}$ . Allow one of the  $2n$  lines to move subject to the condition that the orthocentre of every set of  $2n-1$  lines, chosen from the  $2n$ , shall always lie on this directrix. The problem is to find the curve enveloped by the moving line. Equating the clinant of the directrix to a constant turn  $1/k$ , we have

$$\frac{a_{n+1}}{a_n} = 1/k,$$

whence

$$a_n - k a_{n+1} = 0.$$

But

$$a_n = \sum_{i=1}^{2n} \frac{x_1 t_1^n}{(t_1 - t_2)(t_1 - t_3) \dots (t_1 - t_{2n})},$$

and

$$a_{n+1} = \sum_{i=1}^{2n} \frac{x_1 t_1^{n-1}}{(t_1 - t_2)(t_1 - t_3) \dots (t_1 - t_{2n})}.$$

Then

$$a_n - k a_{n+1} = \sum_{i=1}^{2n} \frac{x_1 (t_1^n - k t_1^{n-1})}{(t_1 - t_2)(t_1 - t_3) \dots (t_1 - t_{2n})} = 0.$$

Considering as the moving line the line whose reflection in the origin is  $x_1$ , and whose clinant is  $-1/t_1$ , this last equation may be written in the form

$$x_1 (t_1^n - k t_1^{n-1}) = (\alpha_1 t_1 + \alpha_2)^{2n-2},$$

for  $x_1$  and  $t_1$  are the only variables.  $\alpha_1$  and  $\alpha_2$  are symbolic, and have meaning only in combination.

Now the condition that the equation of the line in the form taken,

$$x t_a + y = x_a t_a,$$

shall represent a line is that

$$x_a t_a = y_a.$$

Substituting from this for  $t_1$  in the above equation, we obtain

$$x_1^{n-1} y_1^n - k x_1^n y_1^{n-1} = (\alpha_1 y_1 + \alpha_2 x_1)^{2n-2}, \quad (1)$$

which is the pedal of the curve enveloped by the moving line.

The equation of a line in circular coordinates with clinant  $-1/t$  is

$$\frac{x}{x_1} + \frac{y}{y_1} = 1,$$

where  $x_1$  is the reflection of the origin in the moving line, and  $y_1$  is its conjugate. The equation of any line in conjugate coordinates may be written

$$\xi x + \eta y + \zeta = 0,$$

where  $\xi$ ,  $\eta$  and  $\zeta$  may be considered as trilinear line coordinates, having for triangle of reference the triangle whose vertices consist of the  $I$  and  $J$  points on the line at infinity and the origin. In order for these two lines to be identical,

$$\frac{1}{x_1} = -\frac{\xi}{\zeta},$$

$$\frac{1}{y_1} = -\frac{\eta}{\zeta}.$$

Dividing equation (1) through by  $x_1^{2n-2} y_1^{2n-2}$ , and substituting  $-\frac{\xi}{\zeta}$  for  $\frac{1}{x_1}$  and  $-\frac{\eta}{\zeta}$  for  $\frac{1}{y_1}$ , the equation of the envelope of the moving line is obtained in the form

$$\zeta(-\xi^{n-1}\eta^{n-2} + k\xi^{n-2}\eta^{n-1}) = (\alpha_1\xi + \alpha_2\eta)^{2n-2}.$$

This is a curve of class  $2n-2$ , and since the equation has no terms in  $\zeta$  higher than the first, contains the line at infinity as a  $(2n-3)$ -fold line. The  $2n-3$  points of tangency, obtained by equating

$$-\xi^{n-1}\eta^{n-2} + k\xi^{n-2}\eta^{n-1}$$

to zero, are on the line at infinity in the directions  $\xi = 0$   $n-2$  times,  $\eta = 0$   $n-2$  times, and  $\xi = k\eta$  once, where  $1/k$  is the clinant of the directrix. The curve is therefore a special and specially placed  $(2n-3)$ -fold parabola, being tangent to the line at infinity  $n-2$  times at  $I$ ,  $n-2$  times at  $J$ , and once in the direction given by  $\xi = k\eta$ .

Moreover, since the line at infinity enters as a  $(2n-3)$ -fold line, and the equation is of class  $2n-2$ , the curve is a special JONQUIÈRES curve in lines. Giving to this curve the name  $K^{2n-2}$ , we may state generally:

**THEOREM 1.** *Given  $2n$  lines, one of which is allowed to move subject to the condition that the orthocentre of every  $2n-1$  lines chosen from the  $2n$ , shall always lie on the directrix, the resulting curve is a  $K^{2n-2}$ .*

The simplest  $K^{2n-2}$ , obtained when  $n = 2$ , is evidently

$$\zeta(-\xi + k\eta) = (\alpha_1\xi + \alpha_2\eta)^2,$$

which is the equation of a parabola, tangent to the line at infinity in the direction given by  $\xi = k\eta$ . Now, a parabola, which has for the clinant of its directrix  $1/k$ , must have  $-1/k$  for the clinant of the direction to the point where it is tangent to the line at infinity, since this point must lie in a perpendicular direction. That is, in the case of the parabola  $K^2$ , the direction indicated by  $\xi = k\eta$  is the direction whose clinant is  $-1/k$ . But since all curves  $K^{2n-2}$ , including the parabola, are tangent to the line at infinity in the direction given by  $\xi = k\eta$ , this means that *they are all tangent to the line at infinity at the point, the clinant of the direction to which is  $-1/k$ .*

If the original  $2n$  lines are chosen so that  $1/k$  is equal to  $-1$ , the directrix is perpendicular to the axis of reals, and the direction indicated by  $\xi = k\eta$ , in which all curves  $K^{2n-2}$  are tangent to the line at infinity, is the direction of the axis of reals. Hereafter, every  $K^{2n-2}$ , which is tangent to the line at infinity  $n-2$  times at  $I$ ,  $n-2$  times at  $J$ , and *once in the direction of the axis of reals, will be called a  $K_0^{2n-2}$ .*

For  $n = 3$  and  $k = -1$ , we have the general JONQUIÈRES curve  $K_0^4$ , containing the line at infinity as its triple line. Thought of as a triple parabola, the curve is specially placed, being tangent to the line at infinity at  $I$ ,  $J$ , and in the direction of the axis of reals.

*Orthocentre of  $2n-1$  Lines of a  $K^{2n-2}$ .*

Any one of the  $2n$  lines might be allowed to move subject to the condition that the orthocentre of every set of  $2n-1$  lines, chosen from the  $2n$ , should always lie on the fixed directrix, and the resulting curve would be identically the same, for the moving line must at different times coincide with each position of the  $2n-1$  fixed lines, since their orthocentre lies on the directrix. Inasmuch as any one of the  $2n$  lines might be allowed to vary for a time, then remain fixed, and one of the others be allowed to vary, subject to the given conditions, it is evident that the  $2n-1$  lines are any  $2n-1$  lines of the curve. Hence, the theorem follows :

**THEOREM 2.** *The orthocentre of any  $2n-1$  lines of a  $K^{2n-2}$  must always lie on the directrix.*

Thus, in the case of the  $K^4$ , where  $n = 3$  (that is, in the case of the JONQUIÈRES curve of the fourth class, having the line at infinity as a triple line), the orthocentre of *any* five lines of the curve must always lie on its directrix.

§ 2. Some Theorems Concerning the Orthocentre of  $2n-1$  Lines.

In the *Orthocentric Properties*, already mentioned, it was proved that the first orthocentre of  $2n-1$  lines is of the form  $\frac{S_{n-1}}{S_{2n-1}}$ , when the lines are taken as tangents of a  $\Delta^{2n-1}$ . By  $S_{n-1}$  is meant the sum of the products of the  $2n-1$   $t$ 's, taken  $n-1$  at a time, where  $1/t_a$  is the clinant of the line  $\alpha$ , and similarly for  $S_{2n-1}$ . For the work of this section it is more convenient to take the equation of the  $\Delta^{2n-1}$  in the form

$$\begin{aligned} (-)^n (x t^{n-1} - y t^n) = 1 - t^{2n-1} - (S_{2n-2} t - S_1 t^{2n-2}) + \dots \\ + (-)^n (S_{n+1} t^{n-2} - S_{n-2} t^{n+1}), \end{aligned}$$

where the clinant of any line  $\alpha$  is now  $t_a$ . Consequently, the orthocentre of  $2n-1$  lines tangent to a  $\Delta^{2n-1}$  is now  $S_n$ .

Considering three lines tangent to a  $\Delta^3$ , which may be any three lines in the plane, since a  $\Delta^3$  can be taken tangent to any four lines, their orthocentre is  $S_2$ , where  $S$  stands for symmetric functions of the three  $t$ 's, which are the respective clinants of the three lines. This may be written

$$S_2 + t_3 S_1,$$

where  $S$  now stands for symmetric functions of two  $t$ 's. If  $S_1$  vanishes; *i. e.*, if

$$t_1 + t_2 = 0,$$

the orthocentre is evidently independent of the third line, and is

$$-t_1^2.$$

The condition on the two remaining lines, since the clinant of one is equal to the negative of the other, is *that they be perpendicular*. This is nothing more than the well-known case of elementary geometry, where the orthocentre of a right-angled triangle is at the vertex of the right angle.

Considering next five lines tangent to a  $\Delta^5$ , which may be any five lines in the plane, since a  $\Delta^5$  can be taken tangent to any six lines, their orthocentre is  $S_3$ , where  $S$  stands for symmetric functions of the five  $t$ 's, which are the clinants of the five lines. This may be written

$$S_3 + t_5 S_2 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

If  $S_2$  vanishes, the orthocentre is evidently independent of the fifth line, and is

$$S_3 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

This may be written

$$S_3 + t_4 S_2 \quad (S \text{ for } 3 \text{ } t\text{'s}).$$

If  $S_2$  vanishes; that is, if

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = 0,$$

which carries with it its conjugate relation

$$t_1 + t_2 + t_3 = 0,$$

the orthocentre is evidently independent of the fourth line, and is

$$S_3 \quad (S \text{ for 3 } t's);$$

that is,

$$t_1 t_2 t_3.$$

Since  $S_1$  and  $S_2$  for three  $t$ 's vanish, it follows that

$$t_1 = \omega t_2 = \omega^2 t_3,$$

and the orthocentre is

$$t_1^3.$$

That is, the condition on the three remaining lines is *that they form an equilateral triangle*.

Now the equation of any line tangent to a  $\Delta^5$  is

$$1 - at + xt^2 - yt^3 + bt^4 - t^5 = 0.$$

The three intersections of the three lines, which form an equilateral triangle, are:

$$x_1 = -\frac{a}{t_1} + bt_1^2 + t_1^3,$$

$$x_2 = -\frac{a\omega^2}{t_1} + b\omega^2 t_1^2 + t_1^3,$$

$$x_3 = -\frac{a\omega}{t_1} + b\omega t_1^2 + t_1^3.$$

Their centroid is  $t_1^3$ , which is the orthocentre. Therefore, *the orthocentre of any five lines is independent of two of the lines, when the remaining three form an equilateral triangle, and is the same as the centroid of the three points of intersection of the three lines*.

Now consider seven lines tangent to a  $\Delta^7$ , which may be any seven lines in the plane, since a  $\Delta^7$  can be put tangent to any eight lines. Their orthocentre is  $S_4$ , where  $S$  stands for symmetric functions of seven  $t$ 's. This may be written

$$S_4 + t_7 S_3 \quad (S \text{ for 6 } t's).$$

If  $S_3$  vanishes, the orthocentre is independent of the seventh line, and is

$$S_4 \quad (S \text{ for 6 } t's),$$

which may be written

$$S_4 + t_6 S_3 \quad (S \text{ for 5 } t's).$$

If  $S_3$  vanishes, the orthocentre is independent of the sixth line, and is

$$S_4 \quad (S \text{ for } 5 \text{ } t\text{'s}).$$

This may be written

$$S_4 + t_5 S_3 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

If  $S_3$  vanishes, the orthocentre is independent of the fifth line, and is

$$S_4 \quad (S \text{ for } 4 \text{ } t\text{'s});$$

that is,

$$t_1 t_2 t_3 t_4.$$

The vanishing of  $S_3$ , where  $S$  stands for symmetric functions of four  $t$ 's, carries with it its conjugate relation

$$S_1 = 0 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

The vanishing of  $S_3$  for five  $t$ 's may be expressed :

$$S_3 + t_5 S_2 = 0 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

But in order for the orthocentre to be independent of the fifth line, it was proved that  $S_3$  of four  $t$ 's must vanish. Consequently,

$$S_2 = 0 \quad (S \text{ for } 4 \text{ } t\text{'s}).$$

From these three relations,

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = 0 \quad (S \text{ for } 4 \text{ } t\text{'s}),$$

it follows that the four remaining lines form an ordered 4-gon with equal angles  $\frac{\pi}{4}$ , for the relation

$$t_1 = i t_2 = -t_3 = -i t_4$$

exists between their clinants. Consequently, the orthocentre reduces to

$$-t_1^4.$$

The equation of any line tangent to a  $\Delta^7$  is

$$1 - a_2 t + a_1 t^2 - x t^3 + y t^4 - b_1 t^5 + b_2 t^6 - t^7 = 0.$$

The four points of intersection of the four lines which form an ordered 4-gon with equal angles  $\frac{\pi}{4}$ , are :

$$\begin{aligned} x_1 &= \frac{a_2 i}{t_1^2} + \frac{a_1(1-i)}{t_1} + b_1 i t_1^2 - b_2(i-1)t_1^3 - t_1^4, \\ x_2 &= -\frac{a_2 i}{t_1^2} + \frac{a_1(-i-1)}{t_1} - b_1 i t_1^2 - b_2(1+i)t_1^3 - t_1^4, \\ x_3 &= \frac{a_2 i}{t_1^2} + \frac{a_1(-1+i)}{t_1} + b_1 i t_1^2 - b_2(-i+1)t_1^3 - t_1^4, \\ x_4 &= -\frac{a_2 i}{t_1^2} + \frac{a_1(i+1)}{t_1} - b_1 i t_1^2 - b_2(-1-i)t_1^3 - t_1^4. \end{aligned}$$

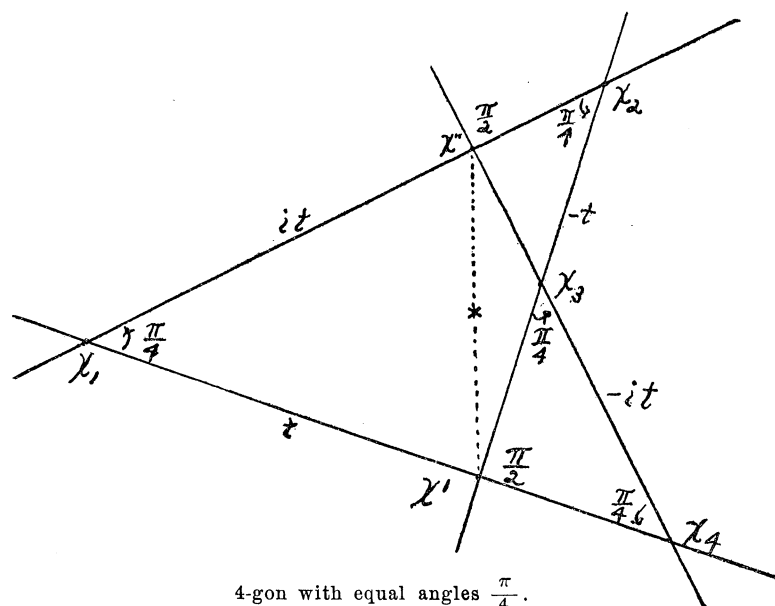
Their centroid is  $-t_1^4$ , which is the orthocentre. Therefore, *the orthocentre of any seven lines is independent of three of the lines, when the four remaining lines form an ordered 4-gon with equal angles  $\frac{\pi}{4}$ , and is the centroid of the four points at which are the angles  $\frac{\pi}{4}$ .*

But in addition to the four  $\frac{\pi}{4}$  points of intersection of the four lines, there are two points where the lines intersect at the angle  $\frac{\pi}{2}$ . The centroid of these two points is also the same as the orthocentre of the seven lines where three are arbitrary, for these points are

$$x' = -\frac{a_2}{t_1^2} - b_1 t_1^2 - t_1^4,$$

$$x'' = \frac{a_2}{t_1^2} + b_1 t_1^2 - t_1^4,$$

and their centroid is  $-t_1^4$ , which is the orthocentre.



4-gon with equal angles  $\frac{\pi}{4}$ .

Passing to the case of nine lines tangent to a  $\Delta^9$ , the orthocentre  $S_5$ , where  $S$  stands for symmetric functions of nine  $t$ 's, is independent of the sixth, seventh, eighth and ninth lines, *when the remaining five lines form an equiangular pentagon, the equal angles being  $\frac{\pi}{5}$ , and becomes  $t_1^5$ .* As in the other cases, this is easily

proved by calculation to be the centroid of the five points at which are the angles  $\frac{\pi}{5}$ .

Now it is evident that the process we have been carrying on can be continued without limit, and that the orthocentre  $S_n$  of  $2n-1$  lines tangent to a  $\Delta^{2n-1}$  is independent of  $n-1$  of the lines, and becomes  $(-1)^{n-1}t^n$ , when the remaining  $n$  lines form an ordered polygon with equal angles  $\frac{\pi}{n}$ , and that the orthocentre is always the centroid of the  $n$   $\frac{\pi}{n}$  points of intersection of these  $n$  lines.

But it was observed in the case of four lines which formed an ordered 4-gon with equal angles  $\frac{\pi}{4}$ , that while the centroid of the four  $\frac{\pi}{4}$  points was the orthocentre of seven lines, where three were arbitrary, the centroid of the two  $\frac{\pi}{2}$  points was also the orthocentre of the seven lines.

Passing to the case of five lines which form an ordered pentagon with equal angles  $\frac{\pi}{5}$ , there are five points where the intersecting lines make the angle  $\frac{\pi}{5}$ , and five points where they make the angle  $\frac{2\pi}{5}$ . The centroids of both these sets of points are the same, as is easily proved by taking the five lines as tangents of a  $\Delta^5$ , and calculating the centroids of the two sets of points.

In the case of six lines which form an ordered hexagon with equal angles  $\frac{\pi}{6}$ , there are six angles  $\frac{\pi}{6}$ , six angles  $\frac{2\pi}{6}$  and three angles  $\frac{3\pi}{6}$ , and the centroid of each set of points is the same, as can most easily be proved by taking the six lines as tangents of a  $\Delta^6$ , and calculating the centroid of each set of points.

The general scheme, for convenience printed at the end of this section, may be written down in the case of  $n$  lines, which form an ordered polygon having equal angles  $\frac{\pi}{n}$ .

The fact, proved in the particular cases, concerning the centroid of the sets of points in the case of  $n$  lines which form an ordered polygon having equal angles  $\frac{\pi}{n}$ , is evidently true in general, and may be verified in any particular case, following the given method of proof. Hence, the theorem follows:

**THEOREM 3.** *The centroid of any one of the sets of points associated with  $n$  lines which form an ordered polygon with equal angles  $\frac{\pi}{n}$ , is the centroid of all.*

Also, since the centroid of the  $n \frac{\pi}{n}$  points was proved to be the orthocentre of  $2n-1$  lines, the theorem follows:

**THEOREM 4.** *The orthocentre of  $2n-1$  lines is independent of  $n-1$  of the lines when the remaining  $n$  lines form an ordered polygon with equal angles  $\frac{\pi}{n}$ , and is the centroid of any set of points associated with the  $n$  lines.*

TABLE.

|          |            |          |        |                      |                                                                                                                                                        |
|----------|------------|----------|--------|----------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| 2        | lines give | 1        | angle  | $\frac{\pi}{2}$      | .                                                                                                                                                      |
| 3        | “ “        | 3        | angles | $\frac{\pi}{3}$      | .                                                                                                                                                      |
| 4        | “ “        | 4        | “      | $\frac{\pi}{4}$ ,    | 2 angles $\frac{2\pi}{4}$ .                                                                                                                            |
| 5        | “ “        | 5        | “      | $\frac{\pi}{5}$ ,    | 5 “ $\frac{2\pi}{5}$ .                                                                                                                                 |
| 6        | “ “        | 6        | “      | $\frac{\pi}{6}$ ,    | 6 “ $\frac{2\pi}{6}$ , 3 angles $\frac{3\pi}{6}$ .                                                                                                     |
| 7        | “ “        | 7        | “      | $\frac{\pi}{7}$ ,    | 7 “ $\frac{2\pi}{7}$ , 7 “ $\frac{3\pi}{7}$ .                                                                                                          |
| 8        | “ “        | 8        | “      | $\frac{\pi}{8}$ ,    | 8 “ $\frac{2\pi}{8}$ , 8 “ $\frac{3\pi}{8}$ , 4 angles $\frac{4\pi}{8}$ .                                                                              |
| 9        | “ “        | 9        | “      | $\frac{\pi}{9}$ ,    | 9 “ $\frac{2\pi}{9}$ , 9 “ $\frac{3\pi}{9}$ , 9 “ $\frac{4\pi}{9}$ .                                                                                   |
| 10       | “ “        | 10       | “      | $\frac{\pi}{10}$ ,   | 10 “ $\frac{2\pi}{10}$ , 10 “ $\frac{3\pi}{10}$ , 10 “ $\frac{4\pi}{10}$ , 5 angles $\frac{5\pi}{10}$ .                                                |
| .....    |            |          |        |                      |                                                                                                                                                        |
| .....    |            |          |        |                      |                                                                                                                                                        |
| .....    |            |          |        |                      |                                                                                                                                                        |
| $2m$     | “ “        | $2m$     | “      | $\frac{\pi}{2m}$ ,   | $2m$ “ $\frac{2\pi}{2m}$ , $2m$ “ $\frac{3\pi}{2m}$ , .....,<br>$2m$ angles $\frac{(m-1)\pi}{2m}$ , $m$ angles $\frac{m\pi}{2m}$ .                     |
| $(2m+1)$ | “ “        | $(2m+1)$ | “      | $\frac{\pi}{2m+1}$ , | $(2m+1)$ “ $\frac{2\pi}{2m+1}$ , $(2m+1)$ “ $\frac{3\pi}{2m+1}$ , .....,<br>$(2m+1)$ “ $\frac{(m-1)\pi}{2m+1}$ , $(2m+1)$ angles $\frac{m\pi}{2m+1}$ . |

### § 3. General Discussion of the $K^{2n}$ .

In the first section, the equation of the  $K^{2n-2}$  was obtained in trilinear complex line coordinates, and the orthocentre of any  $2n-1$  of its lines was proved to lie on the directrix of the curve. In the present section an equation of the curve will be derived by another method, and a general discussion will follow. Some special cases will also be considered.

*Envelope of a  $\Delta^{2n-1}$  Moving Under Parabolic Translation.*

Consider the general case of a  $\Delta^{2n-1}$  moving under parabolic translation; that is, a translation in which one and therefore all the points of the  $\Delta^{2n-1}$  describe a parabola, although not the same parabola. A convenient form in which to write the map equation of the general parabola for the present purpose is

$$x = \frac{\tau^3}{(\tau^2 - t)^2},$$

where  $\tau$  is a constant turn. This parabola is of unit size, but since the unit of measurement may be taken arbitrarily, it may be considered as any parabola. On dividing the map equation by its conjugate, and substituting the value of  $t$  for which  $x$  becomes infinite,  $1/\tau^2$  is found to be the clinant of the vector from the origin to the point at infinity on the parabola.

The map equation of the general  $\Delta^{2n-1}$  with its centre at the origin is

$$x = (-)^n \left[ n a_{n-1} t^{n-1} - (n-1) a_{n-2} t^{n-2} + \dots + (-)^n \left( \dots + 2 a_1 t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2) b_{n-2}}{t^{n-1}} + \frac{(n-1) b_{n-1}}{t^n} \right].$$

The condition for the envelope of the  $\Delta^{2n-1}$ , moving with its centre on the parabola

$$x = \frac{\tau^3}{(\tau^2 - t)^2},$$

while its orientation remains constant, is that

$$\frac{t D_t f}{t_1 D_{t_1} f} = \text{a real quantity} = \frac{\overline{t D_t f}}{t_1 D_{t_1} f}$$

in the equation

$$x = \frac{\tau^3}{(\tau^2 - t_1)^2} + (-)^n \left[ n a_{n-1} t^{n-1} - (n-1) a_{n-2} t^{n-2} + \dots + (-)^n \left( \dots + 2 a_1 t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2) b_{n-2}}{t^{n-1}} + \frac{(n-1) b_{n-1}}{t^n} \right].$$

This condition is satisfied when

$$t = t_1.$$

Accordingly the equation of the envelope of a  $\Delta^{2n-1}$  moving under parabolic translation, is

$$x = \frac{\tau^3}{(\tau^2 - t)^2} + (-)^n \left[ n a_{n-1} t^{n-1} - (n-1) a_{n-2} t^{n-2} + \dots + (-)^n \left( \dots + 2 a_1 t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2) b_{n-2}}{t^{n-1}} + \frac{(n-1) b_{n-1}}{t^n} \right].$$

A vector from the origin to any point on the envelope can therefore be considered as the sum of two vectors, one from the origin to the centre of the  $\Delta^{2n-1}$ , and the other from the centre of the  $\Delta^{2n-1}$ , which is a point of the parabola, to the corresponding point of the  $\Delta^{2n-1}$ .

Since the  $\Delta^{2n-1}$  contains the line at infinity  $n-1$  times as an isolated double line, and since the parabola is tangent to it once in the direction  $1/\tau^2$ , the envelope must contain it as a special  $(2n-1)$ -fold line, and may be thought of as a special and specially placed  $(2n-1)$ -fold parabola, being tangent to the line at infinity  $n-1$  times at  $I$ ,  $n-1$  times at  $J$ , and once in the direction  $1/\tau^2$ . The envelope is of order  $2n+2$ , as is seen on substituting from the map equation in the equation of a line, and it is scarcely necessary to remark, that on account of the singularities at  $I$  and  $J$  in the case of the  $\Delta^{2n-1}$  and consequently in the case of the envelope, at each point the  $n-1$  points of tangency are equivalent to but  $n$  coincident intersections of the curve with the line at infinity.

On multiplying the map equation of the envelope by  $t$  and subtracting the conjugate equation, the relation

$$tx - y - \frac{\tau t}{\tau^2 - t} + (-)^n \left[ \frac{b_{n-1}}{t^{n-1}} - \frac{b_{n-2}}{t^{n-2}} + \dots + (-)^n \left( \dots + \frac{b_1}{t} - a_1 t^2 + \dots \right) \dots + a_{n-2} t^{n-1} - a_{n-1} t^n \right] = 0$$

is obtained. This equation is self-conjugate, and on eliminating  $y$  by partial differentiation with respect to  $t$ , gives again the map equation. This self-conjugate relation is therefore the line equation of the envelope. From this equation, the envelope is seen to be of class  $2n$ , for, clearing of fractions, the parameter  $t$  enters to the  $2n$ th power. Hence, it is perhaps better to think of the envelope as a special JONQUIÈRES curve of class  $2n$ . When it has been proved that the orthocentre of any  $2n+1$  of its lines lies on a directrix whose clinant is  $-1/\tau^2$ , it will further be known to be a  $K^{2n}$ . To introduce the proof of this, we consider some special cases.

#### *Directrices.*

The equation of the directrix of the parabola

$$x = \frac{\tau^3}{(\tau^2 - t)^2},$$

as the locus of two perpendicular lines, is evidently

$$x = \frac{\tau^3}{\tau^4 - t^2}.$$

The line equation of the envelope of a  $\Delta^3$  moving with its centre on this parabola while its orientation remains constant, is

$$tx - y - \frac{t\tau}{\tau^2 - t} + \frac{b_1}{t} - a_1 t^2 = 0.$$

This represents a general JONQUIÈRES curve of the fourth class, or, thought of otherwise, a triple parabola tangent to the line at infinity at  $I, J$ , and in the direction  $1/\tau^2$ . Now, from Theorem 4, § 2, it is known that the orthocentre of any five lines of the curve is independent of two of the lines, when the remaining three form an equilateral triangle, and is the centroid of the three  $\frac{\pi}{3}$  points of intersection of the three lines. The clinant of any line is  $1/t$ . The clinants of any three lines, which form an equilateral triangle, are  $1/t, 1/\omega t, 1/\omega^2 t$ . The three  $\frac{\pi}{3}$  points of intersection of these three lines are :

$$\begin{aligned} x_{12} &= \frac{\tau^3}{(\tau^2 - t)(\tau^2 - \omega t)} + \frac{b}{\omega t^2} + at(1 + \omega), \\ x_{23} &= \frac{\tau^3}{(\tau^2 - \omega t)(\tau^2 - \omega^2 t)} + \frac{b}{t^2} + at(\omega + \omega^2), \\ x_{31} &= \frac{\tau^3}{(\tau^2 - \omega^2 t)(\tau^2 - t)} + \frac{b}{\omega^2 t^2} + at(\omega^2 + 1). \end{aligned}$$

Their centroid is

$$\frac{1}{3}(x_{12} + x_{23} + x_{31}) = \frac{\tau^5}{\tau^6 - t^3} = x, \text{ say.}$$

This then is the orthocentre. Allowing  $t^3$  to vary,  $x$  evidently traces out a line, which is the locus of the orthocentres for every five lines of the curve, and which is therefore the directrix of the curve. The clinant of the directrix is the clinant of the direction of the point at infinity on it. The point at infinity is obtained from the equation of the directrix when  $t^3 = \tau^6$ . Hence, substituting this value for  $t^3$  in the expression

$$\frac{x}{y} = \frac{\frac{\tau^5}{\tau^6 - t^3}}{\frac{\tau^6 - t^3}{\tau^6 - t^3}} = -\frac{\tau^4}{t^3},$$

the clinant of the directrix is found to be

$$-\frac{1}{\tau^2}.$$

Now, the line equation of the envelope of the parabola and the  $\Delta^5$  is

$$tx - y - \frac{\tau t}{\tau^2 - t} - \frac{b_2}{t^2} + \frac{b_1}{t} - a_1 t^2 + a_2 t^3 = 0.$$

This represents a special JONQUIÈRES curve of class six, being tangent to the line at infinity twice at  $I$ , twice at  $J$ , and once in the direction  $1/\tau^2$ . The two points of tangency at  $I$ , and likewise at  $J$ , are equivalent to but three coincident intersections of the curve with the line at infinity.

Now, from Theorem 4, § 2, it is known that the orthocentre of any seven lines of the curve is independent of three of the lines when the remaining four form an ordered 4-gon with equal angles  $\frac{\pi}{4}$ , and is the centroid of the two  $\frac{\pi}{2}$  points of intersection of the four lines. The clinants of any four lines of the curve which form an ordered 4-gon with equal angles  $\frac{\pi}{4}$  are  $1/t$ ,  $1/it$ ,  $-1/t$ ,  $-1/it$ . One point  $\frac{\pi}{2}$  is given by the intersection of the two lines whose clinants are  $1/t$  and  $-1/t$ . The other  $\frac{\pi}{2}$  point is given by the intersection of the other two lines, whose clinants are  $1/it$  and  $-1/it$ . These two points are

$$x_{13} = \frac{\tau^3}{(\tau^2 - t)(\tau^2 + t)} - \frac{b_1}{t^2} - a_2 t^2,$$

$$x_{24} = \frac{\tau^3}{(\tau^2 - it)(\tau^2 + it)} + \frac{b_1}{t^2} + a_2 t^2.$$

Their centroid is

$$x = \frac{\tau^7}{\tau^8 - t^4}.$$

This, then, is the orthocentre. Allowing  $t^4$  to vary,  $x$  evidently traces out a line, which is the locus of the orthocentres of every seven lines of the curve. The clinant of this directrix, found as before, is  $-1/\tau^2$ .

Similarly, the locus of the orthocentre for any nine lines of the envelope of the parabola and a  $\Delta^7$  is

$$x = \frac{\tau^9}{\tau^{10} - t^5},$$

and its clinant is  $-1/\tau^2$ .

From these equations, it is evident that the orthocentre of any  $2n + 1$  lines of the envelope of the parabola and a  $\Delta^{2n-1}$  lies on a directrix, whose equation is

$$x = \frac{\tau^{2n+1}}{\tau^{2n+2} - t^{n+1}}.$$

Its clinant, calculated as above, is  $-1/\tau^2$ . Since the equations of the directrices all give the same clinant  $-1/\tau^2$ , and since the absolute value of the reflection of the origin in them, being  $|\tau|$ , is equal to unity, it is evident that they are all the same. Hence:

**THEOREM 5.** *The envelope of a  $\Delta^{2n-1}$  moving under parabolic translation is a  $K^{2n}$ , whose directrix is the same as the directrix of the parabola described by the centre of the  $\Delta^{2n-1}$ .*

#### *Method of Plotting the $K^{2n}$ .*

The line equation of the general parabola used above is

$$tx - y - \frac{t\tau}{\tau^2 - t} = 0.$$

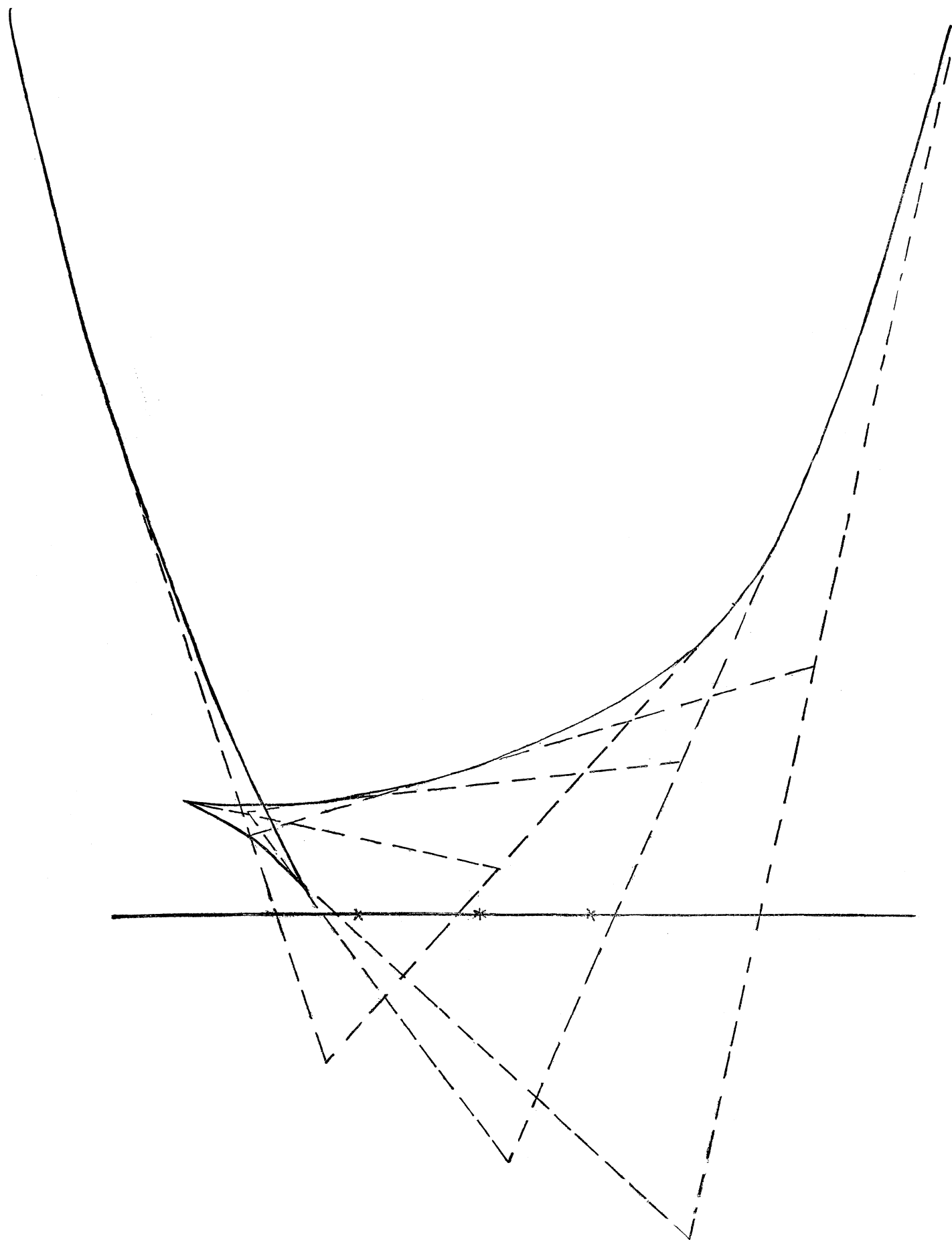
The line equation of the  $\Delta^{2n-1}$  is

$$\begin{aligned} (-)^n (xt^n - yt^{n-1}) = t^{2n-1} - 1 - (S_1 t^{2n-2} - S_{2n-2} t) + \dots \\ + (-)^n (S_{n-2} t^{n+1} - S_{n+1} t^{n-2}). \end{aligned}$$

The line equation of the  $K^{2n}$  is

$$\begin{aligned} tx - y - \frac{\tau t}{\tau^2 - t} + (-)^n \left[ \frac{b_{n-1}}{t^{n-1}} - \frac{b_{n-2}}{t^{n-2}} + \dots + (-)^n \left( \dots + \frac{b_1}{t} - a_1 t^2 + \dots \right) \right. \\ \left. \dots + a_{n-2} t^{n-1} - a_{n-1} t^n \right] = 0. \end{aligned}$$

The tangents at corresponding points of these three curves are parallel, for the clinant of each is the same,  $1/t$ . Since it was proved that a vector from the origin to any point on the envelope of the parabola and the  $\Delta^{2n-1}$  is equal to the sum of the vector from the origin to the centre of the  $\Delta^{2n-1}$ , and of the vector from the centre of the  $\Delta^{2n-1}$ , which is a point on the parabola, to the corresponding point on the  $\Delta^{2n-1}$ , it is evident that by drawing a tangent to the  $\Delta^{2n-1}$  which is parallel to the tangent to the parabola at the centre of the  $\Delta^{2n-1}$ , a corresponding point of the  $\Delta^{2n-1}$  is thus obtained, which is a point of the  $K^{2n}$ . As the centre of the  $\Delta^{2n-1}$  takes all positions on the parabola, while its orientation remains constant, all points of the  $K^{2n}$  are thus obtained.



$K_0^4$  with two real cusps, and its directrix.

Consider now the more general question of the envelope of a  $\Delta^{2m-1}$ , whose orientation is constant, and whose centre is moving on a  $K^{2n}$ . The condition for the envelope is that

$$\frac{t D_t f}{t_1 D_{t_1} f} = \text{its conjugate}$$

in the equation

$$\begin{aligned} x = & \frac{\tau^3}{(\tau^2 - t)^2} + (-)^n \left[ n a_{n-1} t^{n-1} - (n-1) a_{n-2} t^{n-2} + \dots \right. \\ & + (-)^n \left( \dots + 2a_1 t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2) b_{n-2}}{t^{n-1}} + \frac{(n-1) b_{n-1}}{t^n} \Big] \\ & + (-)^m \left[ m a'_{m-1} t_1^{m-1} - (m-1) a'_{m-2} t_1^{m-2} + \dots \right. \\ & + (-)^m \left( \dots + 2a'_1 t_1 + \frac{b'_1}{t_1^2} - \dots \right) \dots - \frac{(m-2) b'_{m-2}}{t_1^{m-1}} + \frac{(m-1) b'_{m-1}}{t_1^m} \Big]. \end{aligned}$$

This condition is satisfied, as before, when

$$t = t_1.$$

That is, the equation of the envelope is of the form

$$x = \frac{\tau^3}{(\tau^2 - t)^2} + \Delta^{2n-1} + \Delta^{2m-1}.$$

It is evident, therefore, that when  $m < n$ , the equation reduces simply to

$$x = \frac{\tau^3}{(\tau^2 - t)^2} + \Delta_1^{2n-1},$$

and is another  $K^{2n}$ , with the same directrix as the  $K^{2n}$  over which it was translated, for the equation of the directrix is independent of the constants of the deltoidal term.

When  $m > n$ , the equation of the envelope reduces to

$$x = \frac{\tau^3}{(\tau^2 - t)^2} + \Delta_1^{2m-1},$$

and is evidently a  $K^{2m}$  with the same directrix as the  $K^{2n}$  over which it was translated, since both curves can be obtained by translating the proper deltoid over the same parabola, and, consequently, according to Theorem 5, have the same directrix. Hence, we may state the theorem:

**THEOREM 6.** *The envelope of a  $\Delta^{2m-1}$ , whose orientation is constant, and whose centre moves on a  $K^{2n}$ , is another  $K^{2n}$  if  $m < n$ , and is a  $K^{2m}$  if  $m > n$ , and the directrices of all curves generated thus are the same; namely, the directrix of the  $K^{2n}$ .*

It should be observed that any curve  $K^{2n+2}$  can be obtained by translating a cycloid

$$(n+1)a_n t^n + \frac{nb_n}{t^{n+1}}$$

over a  $K^{2n}$ .

Now, in the equation of the  $K^{2n}$ , if  $1/\tau^2$  be taken equal to unity, the directrix is perpendicular to the axis of reals, and the curve is tangent to the line at infinity  $n-1$  times at  $I$ ,  $n-1$  times at  $J$ , and *once in the direction of the axis of reals*, and is then designated by  $K_0^{2n}$ , according to the notation of § 1. The equation then takes the form

$$x = \frac{1}{(1-t)^2} + (-)^n \left[ na_{n-1} t^{n-1} - (n-1)a_{n-2} t^{n-2} + \dots \right. \\ \left. + (-)^n \left( \dots + 2a_1 t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2)b_{n-2}}{t^{n-1}} + \frac{(n-1)b_{n-1}}{t^n} \right].$$

#### § 4. Caustics.\*

Any rational curve can be represented by a map equation

$$x = f(t),$$

which of course carries with it its conjugate equation

$$y = \bar{f}(t),$$

and the general form of its caustic by reflection, where the incident rays are parallel to the real axis, can easily be derived. Letting  $t^z$  represent the unknown clinant of the line tangent to the curve at the point

$$f(t),$$

the equation of this line can be written

$$x - f(t) = t^z [y - \bar{f}(t)],$$

which, for varying  $t$ , is the line equation of the curve. On dividing through by  $t^z$ , and eliminating  $y$  by partial differentiation with respect to  $t$ , the equation

$$-\frac{zx}{t^{z+1}} - \frac{f'(t)}{t^z} + \frac{zf(t)}{t^{z+1}} = -\bar{f}'(t)$$

or

$$x = f(t) - \frac{tf'(t)}{z} + \frac{t^{z+1}\bar{f}'(t)}{z}$$

is obtained, which must be the map equation of the curve. But, by hypothesis,

$$x = f(t)$$

---

\* Salmon: *Higher Plane Curves*. Heath: *Geometrical Optics*. F. Morley: *On the Caustic of the Epicycloid*.

is the map equation of the curve. It follows, therefore, that

$$\frac{tf'(t)}{z} = \frac{t^{z+1}\bar{f}'(t)}{z},$$

whence

$$t^z = \frac{f'(t)}{\bar{f}'(t)}.$$

This then is the clinant of the tangent to the curve at any point given by  $t$ . Since when the incident rays are parallel to the real axis, the reflected rays make twice the angle with the axis of reals that the tangents to the curve at the points of reflection make with the axis of reals, it is evident that the clinant of any ray after reflection at the point given by  $t$ , is

$$\left[ \frac{f't}{\bar{f}'t} \right]^2,$$

and its equation is

$$x - f(t) = \left[ \frac{f'(t)}{\bar{f}'(t)} \right]^2 [y - \bar{f}(t)],$$

which, as  $t$  varies, is the line equation of the caustic of the given curve. Since the condition for the cusps of the curve

$$x = f(t),$$

obtained by equating  $\frac{\partial x}{\partial t}$  to zero, is

$$f'(t) = 0,$$

which of course carries with it its conjugate equation

$$\bar{f}'(t) = 0,$$

it is evident further that:

**THEOREM 7.** *The cusps of any rational curve lie on its caustic.*

Now, consider the case of the  $K_0^{2n}$ . The condition for cusps is of degree  $2n + 2$ , showing that the  $K_0^{2n}$  has  $2n + 2$  cusps, although not necessarily all real. The line equation of the caustic by reflection, formed as in the general case, is

$$\begin{aligned} x - \left\{ \frac{1}{(1-t)^2} + (-)^n \left[ na_{n-1}t^{n-1} - (n-1)a_{n-2}t^{n-2} + \dots \right. \right. \\ \left. \left. + (-)^n \left( \dots + 2a_1t + \frac{b_1}{t^2} - \dots \right) \dots - \frac{(n-2)b_{n-2}}{t^{n-1}} + \frac{(n-1)b_{n-1}}{t^n} \right] \right\} \\ = \frac{1}{t^2} \left( y - \left\{ \frac{t^2}{(1-t)^2} + (-)^n \left[ \frac{nb_{n-1}}{t^{n-1}} - \frac{(n-1)b_{n-2}}{t^{n-2}} + \dots \right. \right. \right. \\ \left. \left. + (-)^n \left( \dots + \frac{2b_1}{t} + a_1t^2 - \dots \right) \dots - (n-2)a_{n-2}t^{n-1} \right. \right. \\ \left. \left. \left. + (n-1)a_{n-1}t^n \right] \right\} \right), \end{aligned}$$

which may be written in the form

$$\begin{aligned} t^2 x - y = & (-)^n \left[ na_{n-1} t^{n+1} - (n-1) (a_{n-2} + a_{n-1}) t^n \right. \\ & + (n-2) (a_{n-3} + a_{n-2}) t^{n-1} - \dots + (-)^n \left\{ \dots + 2t^3 (a_1 + a_2) \right. \\ & + b_1 - \frac{2}{t} (b_1 + b_2) + \dots \left. \right\} \dots - \frac{(n-2)}{t^{n-3}} (b_{n-3} + b_{n-2}) \\ & + \frac{(n-1)}{t^{n-2}} (b_{n-2} + b_{n-1}) - \frac{nb_{n-1}}{t^{n-1}} \left. \right] - a_1 t^2. \end{aligned}$$

Its map equation is

$$\begin{aligned} x = & (-)^n \left[ \frac{n(n+1)}{2} a_{n-1} t^{n-1} - \frac{(n-1)n}{2} (a_{n-2} + a_{n-1}) t^{n-2} \right. \\ & + \frac{(n-2)(n-1)}{2} (a_{n-3} + a_{n-2}) t^{n-3} - \dots + (-)^n \left\{ \dots + 3t (a_1 + a_2) \right. \\ & + (b_1 + b_2) \frac{1}{t^3} - \dots \left. \right\} \dots + \frac{(n-2)(n-3)}{2t^{n-1}} (b_{n-3} + b_{n-2}) \\ & - \frac{(n-1)(n-2)}{2t^n} (b_{n-2} + b_{n-1}) + \frac{n(n-1)}{2t^{n+1}} b_{n-1} \left. \right] - a_1, \end{aligned}$$

showing that the caustic is a cycloidal curve. It is to be observed, that in forming the equation of the caustic of the  $K_0^{2n}$ , since the parabolic term,  $\frac{1}{(1-t)^2}$ , appeared on the left side of the equation, and its conjugate,  $\frac{t^2}{(1-t)^2}$ , appeared on the right multiplied by  $1/t^2$ , these terms canceled each other. Hence, since the  $K_0^{2n}$  is made up only of a parabolic term and a deltoidal term of class  $2n-1$ , it is evident that *the caustic by reflection of the general  $\Delta^{2n-1}$  with centre at the origin is the same as the caustic by reflection of the  $K_0^{2n}$ , and is the cycloidal curve on which lie the  $2n-1$  cusps of the  $\Delta^{2n-1}$  and the  $2n+2$  cusps of the  $K_0^{2n}$ .*

In general, given any self-conjugate equation, which is a polynomial in  $t$ , thus:

$$\begin{aligned} C: a_1 t^n - a_2 t^{n-1} + a_3 t^{n-2} - \dots + (-)^k (\dots - a_k t^{n-k+1} + x t^{n-k} \\ - a_{k+2} t^{n-k-1} + \dots) \dots + (-)^{n-k} (\dots - b_{k+2} t^{k+1} + y t^k - b_k t^{k-1} \\ + \dots) \dots + (-)^n (\dots + b_3 t^2 - b_2 t + b_1) = 0, \end{aligned}$$

where  $a$  and  $b$  are conjugates, it represents a cyclogen, the clinant of any one of whose lines is  $(-)^n 1/t^{n-2k}$ . Considering the incident rays as coming from the point at infinity on the axis of reals, the clinant of the line equation of the caustic of any  $C_n$  is therefore  $1/t^{2(n-2k)}$ . On writing out a few caustic equations

for the less complicated cyclogens, it is readily observed, that the caustic equations also always represent cyclogens, and that if  $y$  is the coefficient of  $t^k$  in the  $C_n$ , it is also the coefficient of  $t^k$  in the caustic. Further, considering the caustic as a polynomial in  $t$ , it is observed that the middle term is always absent, and that the class of the caustic is equal to the sum of the order of the  $C_n$  and of the order of  $t$  in its clinant; that is, the resulting caustic is of degree  $2n - 2k$  in  $t$ . Hence, the general form of the caustic of the general  $C_n$  may be written, thus:

$$\alpha_1 t^{2n-2k} - \alpha_2 t^{2n-2k-1} + \alpha_3 t^{2n-2k-2} - \dots + (-)^k \left(n - k - \frac{k}{2}\right) (x + \alpha_{k+1}) t^{2n-3k} + \dots \\ + 0 t^{n-k} + \dots + (-)^k \left(n - k - \frac{k}{2}\right) (y + \beta_{k+1}) t^k + \dots + \beta_3 t^2 - \beta_2 t + \beta_1 = 0,$$

where  $\alpha$  and  $\beta$  are conjugates.

### § 5. Discussion of the $K_0^4$ .

#### (a) Cusps.

The map equation of the  $K_0^4$  is

$$x = \frac{1}{(1-t)^2} + 2at + \frac{b}{t^2}.$$

The condition for cusps, obtained by equating  $\frac{\partial x}{\partial t}$  to zero, was found to be

$$\frac{1}{(1-t)^3} + a - \frac{b}{t^3} = 0, \quad (C)$$

which is a sextic in  $t$ , showing the  $K^4$  has six cusps, although not necessarily all real. There are just as many real cusps as equation (C) has roots  $t_i$  which are real turns.

Considering  $a$  and  $b$  as variables, and calling them  $x$  and  $y$ , equation (C) may be written

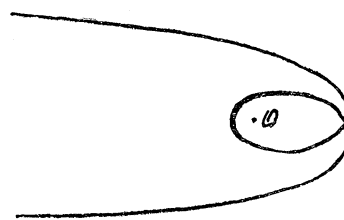
$$\frac{1}{(1-t)^3} + x - \frac{y}{t^3} = 0.$$

For varying  $t$ , this is the line equation of the curve whose map equation is

$$x = -\frac{1}{(1-t)^4}. \quad (A)$$

This curve, composed of two loops, is tangent to the line at infinity in the direction of the real axis, has a double point at  $x = 1/4$ , cuts the axis of reals again at the point  $x = -1/16$ , and is a sextic in lines. Now  $a$  is any point in the plane, and for any point  $a$  there are as many  $t$ 's, which are real turns, satisfying equation (C), as there can be drawn real tangents to the curve (A) from the point  $a$ . It is evident geometrically that if

$$a < -\frac{1}{(1-t)^4},$$



where  $t$  may take any value from  $i$  to  $-i$  (that is, if  $a$  lies inside both loops), no real tangents can be drawn to (A), and the  $K_0^4$  has *no real cusps*. But if

$$a = x = -\frac{1}{(1-t)^4},$$

where  $t$  may take any value from  $i$  to  $-i$  (that is, if  $a$  lies on the smaller loop), two real coincident tangents can be drawn to (A), and the  $K_0^4$  has *two real coincident cusps*. If

$$a > -\frac{1}{(1-t)^4},$$

where  $t$  may take any value from  $i$  to  $-i$ ; and if at the same time

$$a < -\frac{1}{(1-t)^4}$$

where  $t$  may take any value from  $1$  to  $i$  and from  $-i$  to  $1$  (that is, if  $a$  lies outside the smaller loop, but inside the larger), two real distinct tangents can be drawn to (A), and the  $K_0^4$  has *two distinct real cusps*. If

$$a = x = -\frac{1}{(1-t)^4},$$

where  $t$  may take any value from  $1$  to  $i$  and from  $-i$  to  $1$  (that is, if  $a$  lies on the larger loop), two real distinct tangents and two real coincident tangents can be drawn to (A), and the  $K_0^4$  has *two distinct real cusps and two coincident real cusps*. If

$$a = x = 1/4,$$

that is, if  $a$  lies on the node of (A), two distinct pairs of coincident tangents can be drawn to  $a$ , and the  $K_0^4$  has *two distinct pairs of real coincident cusps*. If

$$a > -\frac{1}{(1-t)^4},$$

where  $t$  may take all values (that is, if  $a$  lies outside both loops), four real tangents can be drawn to  $a$ , and the  $K_0^4$  has *four distinct real cusps*. Consequently, it is impossible for the  $K_0^4$  to have more than *four real cusps*.

(b) *Conic on the Cusps.*

The line equation of the  $K_0^4$  may be written as a polynomial in  $t$ , thus:

$$at^4 - t^3(x+a) + t^2(x+y-1) - t(y+b) + b = 0.$$

The invariant  $g_2$  of this quartic in  $t$  is

$$ab - \frac{(x+a)(y+b)}{4} + \frac{(x+y-1)^2}{12},$$

whose vanishing expresses the condition that four  $t$ 's be self-apolar. Hence, the clinants of the four tangents to the  $K_0^4$  from any point on the conic

$$ab - \frac{(x+a)(y+b)}{4} + \frac{(x+y-1)^2}{12} = 0$$

